 Parametric representation

| Syllabus elements | Teaching ideas |
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| Key idea: parametric representation | It is important to explain the ‘big idea’ of parametric representation, and why we use it. Some suggestions are given below: |
| The Concept of a Parameter: Hidden Variables | There is a correlation between IQ and shoe size. In, fact the reason they are correlated is due to another factor: the age of the person.One parameter controls two variables: |
| Recall: | Students have seen Cartesian and parametric representation of the same object before: the circle  and  |
| Why use parametric representation? | For real world modelling, the parametric form may reveal the underlying causes of what we observe more clearly:The parametric form often expresses the locus idea of “a point following a path” more clearly. It is often a simpler way to work with the mathematics.In equations of motion: The parametric form allows us to connect space and time. |
| A fourth form of representation | Students have now learnt another way to represent function:In 2-Unit the function can be expressed in a table, a graph and a Cartesian equationAnd now in Extension 1: in parametric form. |
| Language translation | “Find the Cartesian form” means “eliminate the parameter” |
| Multiple parametric representations | There are typically many ways to convert a Cartesian representation into a parametric one - which is why students aren’t normally asked to do this.The most trivial parametric representation of  would be, !Students should be reminded of this later in the topic so they don’t rely on always seeing the standard parametric form of the parabola. |
| More explorations of Parametric Equations |  An excellent video from James Tanton. 17 minutes long so you may prefer to give this as a homework activity. Will be of particular appeal to students with programming or engineering interestsResources –YouTube – [A Brief Introduction to Parametric Equations](http://www.youtube.com/watch?v=Is3TfStdgUQ) (Tanton Mathematics) – 17 mins |
| The first big challenge for Extension 1 students | This topic is one of the most challenging Preliminary topics in the Extension 1 course. For many students this topic is a dry and intimidating experience which will lead some to dropping Extension 1.Commencing this topic with a structured exploration in GeoGebra will provide a stimulating entry to the topic. This is much more engaging than teaching four or five periods of solid algebra with barely a diagram in sight.Students should be encouraged to see the startling beauty of some of the results - which are truly elegant surprising (see below). |
| Exploring the parametric equations for the parabola | Using GeoGebra, students can explore how the parametric representation works and then discover most of the key relationships they will later develop algebraically. Exploring the role of the parameter: Explore what happens to a point  as the parameter  is changed. Students explore what happens as  varies from negative values to positive values, when . This helps students gain a deep understanding of the role of the parameter and the dynamic nature of the representation (as a locus of a moving point).Exploring the gradient of the tangent at P: Students can add a tangent at P and see if they can find a relationship between the  value and the gradient of the tangent.Explore interacting with a second point Q controlled by a second parameter: Student can add a second point  This helps students clearly see the two parameters are independent, preparing them for harder locus questions later on.Explore the chords joining P and Q: Explore the interactions between the two points and the resulting chord. Challenge students to discover a formula for the gradient of the chord PQ. Then challenge them to arrange PQ in several ways to make different focal chords, and then explore the relationship between  and . Add a directrix to explore the relationship between the tangents at the end points of the focal chord.Throughout the exploration, keep asking students “why?” - how does this relate to what they know about the algebra and the locus so far.In subsequent lessons, students will derive the relationships algebraically which can then be related back to their geometric experiences in the earlier lessons. |
| Geogebra tip: Parametric Representation | Use the Curve command. e.g. |
| Further exploration of Parametric Equations | A video from James Tanton explores the ideas and reasons for using parametric equations, with a focus on parametric equation for a line. Will appeal especially to students with a computing or engineering interest. 17 minutes long, so perhaps best as a homework activity. Good for a flipped lesson.ResourcesYouTube – [A Brief Introduction to Parametric Equations](http://www.youtube.com/watch?v=Is3TfStdgUQ) (Tanton Mathematics) |
| Extension: Translations of the Parametric Form | With a solid understanding of translation and parametric representations, students should be able to deduce parametric equations for translated and reflected parabolas (vertex not at the origin). |
| Highlight the fundamental relationships | There are several key (and surprisingly elegant) results:* The gradient of the tangent is the *same* value as the parameter (and it doesn’t matter what the focal length is!)
* The gradient of the chord is the *average* of the two associated parameter
* When chord is a focal chord,.
* Using the fact that the gradient of the tangents at either endpoint of the chords are  and , it is then clear the tangents on the focal chord are perpendicular. It only takes a little more work to see they meet at the directrix.
 |
| Highlight the beautiful symmetry in the results | Observe how all the equations involving two parameters are symmetrical: you can swap the  and  and you still have the same equation. Why does this *have* to be true? (The equations don’t ‘favour’ one point over the other). |
| Highlight the relationship between the equation for the tangent and the equation for the chord | Observe that the equation for the tangent is the limiting case for the equation of the chord as . This is a very quick way of finding the equation of tangent if you already have the equation of the chord. |
| Highlight the amazing result for the Cartesian equation of the tangent and the chord of contact. | Is the same for both the tangent and the chord of contact. Note however the point  is in different locations: for the tangent the point is on the curve, for the chord of contact the point is outside the curve.Observe also this equation is related toWhy does this have to be true for the tangent equation? (Because the point is on the curve!) |
| Do not memorise the equations | It is important to remind students the derived equations are only true for the *standard* parametrisation . If a different equation is used, the formulas will be different. This has happened in past HSC questions. |
| The Reflection Property | A good hook for this topic is a toy made of two parabolic mirrors. Original version is the Mirage Opti-Gone. Other versions such as 3D-Mirascope available online. |
| Putting the “reflection” back into the Reflection Property | Students may know the reflection property from science:Angle of incidence = Angle of reflection.However they have typically only seen this rule for *flat* surfaces.In the Extension 1 syllabus, the reflection property is described in a different form, which may confuse the science student.The important idea is seen in the diagram to the left. In order for the reflection property to work as required, we want the two angles on either side of the normal to the point to be equal. As a consequence of this, we want the angle between the tangent and the axis to be the same as the angle between the tangent and the focus (using corresponding angles on the parallel lines formed by the axis and by the incoming parallel beam of light). |
| Show this isosceles triangle (and the rhombus) | Most proofs of the reflection property, as scaffolded in HSC exam, rely on proving the existence of the isosceles triangle (or rhombus). It’s important to clearly show this shape during the exploration of the property. |
| “Look Ma, no algebra” | A beautiful geometric proof of the reflection property of the parabola, without the need for any algebra.A Geometric Proof* The locus definition of the parabola tells us that , so  is isosceles with two equal angles  as shown.
* Construct the perpendicular bisector , which gives two equal angles  and in the two triangles.
* Vertically opposite angles gives us.
* Assuming  is the tangent to the parabola at point .
* We are done if we can prove  is the tangent at .

To prove the bisector is indeed the tangent:* The bisector  divides all points in the triangle into two regions: those closer to , and those closer to .
* Imagine a point  on parabola within the triangle. Construct the perpendicular distance to the directrix . By definition,
* Now  is always less than  , so  is always less than  and so  cannot be in the lower half of the triangle.
* So  is the only point on the parabola also on , thus  is the tangent.

So the reflection property is built into our definition of the locus of the parabola.A diagram showing the locus definition of a parabola. Resources – [Focal Properties of Parabola webpage](http://www.cut-the-knot.org/Curriculum/Geometry/ParabolaFocal.shtml) |
| The Square Kilometre Array Project | Your students could be working on this project when they graduate.* •1 square km of radio telescopes, including several thousand parabolic dishes.
* •To be built in South Africa, Australia and New Zealand in radio-quiet remote areas.
* •Construction starting in **2016**, data collection in **2019**.
* •Budget: €1.5 billion
* •Will survey the sky 10,000 times faster than before.
* •Massive computing centres being built to process the data.
* The data from each telescope is combined algorithmically to create the net effect of a single telescope.
* •SKA will generate more data than the entire amount of traffic currently on the internet.
* •Will help test theory of general relativity, provide 3D pictures of ripples at the time of the creation of the universe and may help measure dark matter

Resources – [Wikipedia page on Square Kilometre Array](http://en.wikipedia.org/wiki/Square_Kilometre_Array) |
| Teaching strategies | Strategies to help approach locus problems include:* Use interactive tools to visualise and understand problems,
* Recognise there are two standard types of locus problems: one parameter problems and two parameter problems - start with one parameter problems first.
* Learn some of the classic “algebra tricks” by doing problems and reading worked solutions. Completing the square is an essential skill.

Let students know that HSC questions on parametrics are usually very well scaffolded. |
| Use GeoGebra to observe locus | Example to left: the midpoint  of the chord formed by connecting the point  and  . Easy to construct in Geogebra, then put a trace on the midpoint to see the resulting locus of point .If students have difficulty understanding or working with a specific locus question, it is often helpful to construct the problem in Geogebra and watch the resulting locus. Note that in some cases, the Geogebra construction may require different techniques to achieve the desired locus constraint. |
| Mantra: “Find the locus = eliminate the parameter” | Once the problem is set up, the challenge is to eliminate the parameters without getting bogged down in pages of tortuous algebra. Typically the most efficient method involves rewriting the expressions in terms of , which then allows for simpler substitutions. |
| Just for fun | Putting locus to good work: using the parabola as a multiplication tool.Resources:YouTube – [Mathologer: Multiplying monkeys and parabolic primes](https://www.youtube.com/watch?v=ghHHiGdB-0w) |